David Foster Wallace gives an account of the uncountable

Few ideas have had a racier history than the idea of infinity. It arose amid ancient paradoxes, proceeded to baffle philosophers for a couple of millennia, and then, by a daring feat of intellect, was finally made to yield its secrets in the late nineteenth century—though not without leaving a new batch of paradoxes. You don’t need any specialized knowledge to follow the plot: the main discoveries, despite the ingenuity behind them, can be conveyed with a few strokes of a pen on a cocktail napkin. All of this makes infinity irresistible meat for the popularizer, and quite a few books in that vein have appeared over the years. Now, in “Everything and More: A Compact History of infinity” (Norton; $23.95), the celebrated author David Foster Wallace has set out to initiate readers into its mysteries.

It might seem odd that finite beings like us could come to know anything about infinity, given that we have no direct experience of it. Descartes thought that the idea of infinity was innate, but the behavior of children suggests otherwise; in one study, children in their early school years “reported ‘counting and counting’ in an attempt to find the last number, concluding there was none after much effort.” As it happens, the man who did the most to capture infinity in a theory claimed that his insights were vouchsafed to him by God and ended his life in a mental asylum.

Broadly speaking, there are two versions of infinity. The woollier, more mystical one, which might be called metaphysical infinity, is associated with ideas like perfection, the absolute, and God. The more hard-headed version, mathematical infinity, is the one that Wallace’s book is concerned with. It derives from the idea of endlessness: numbers that can be generated inexhaustibly, time that goes on forever, space that can be subdivided without limit. While metaphysical infinity tends to evoke awe in those who contemplate it, mathematical infinity has, for most of Western intellectual history, been an object of grave suspicion, even scorn. It first cropped up in the fifth century B.C., in the paradoxes of Zeno of Elea. If space is infinitely divisible, Zeno argued, then swift Achilles could never overtake the tortoise: each time he caught up to where the tortoise was, it would have advanced a little farther, ad infinitum. So traumatizing were such paradoxes that Aristotle was moved to ban the idea of a “completed” infinity from Greek thought, setting the orthodoxy for the next two thousand years.

The eventual rehabilitation of the infinite had its origins in another paradox, this one framed by Galileo in 1638. Consider, Galileo said, all the whole numbers: 1, 2, 3, 4, and so on. Now consider just those numbers which are squares: 1, 4, 9, 16, and so on. Surely there are more numbers than squares, since the squares make up just a part of the numbers, and a small part at that. Yet, Galileo observed, there is a way of pairing off squares with whole numbers: 1 to 1, 2 to 4, 3 to 9, 4 to 16, and so on. When two finite sets can be made to correspond in this way, so that each item from the first set gets matched with precisely one item from the second set, and vice versa, you know that they are of equal size without having to go through the tedious business of counting. (In a non-polygamous society, for example, you don’t need to consult the census to know that the number of husbands is the same as the number of wives.) Extending the principle to infinite collections, Galileo felt himself drawn toward the conclusion that there are as many square numbers as there are numbers, period. The part, in other words, was equal to the whole—a result that struck him as absurd.

Two and a half centuries later, Georg Cantor made Galileo’s paradox a basis for a mathematical theory of infinity. Cantor, who lived from 1845 to 1918, was a Russian-born German mathematician with an artistic streak and a keen interest in theology. He saw that the collapse of the familiar logic of part and whole yielded a new definition of infinity, one that did not rely on the vague notion of endlessness. An infinite set, as Cantor characterized it, is one that is the same size as some of its parts—a set, then, that can lose some of its members without being diminished. Now Cantor was in a position to ask a novel question: Are all infinities the same, or are some more infinite than others?

Looking for an infinity bigger than that of the whole numbers, he started by considering the set of fractions. It seemed a good bet, since fractions are densely ordered
on the number line: between every two whole numbers there are infinitely many fractions. (Between 0 and 1, for example, lie 1/2, 1/3, 1/4, 1/5, and so on.) Yet Cantor, to his surprise, was able to find an easy way of matching up the whole numbers and the fractions one to one. Despite initial appearances, these two infinities turned out to be the same. Perhaps, he thought, all infinite sets, by dint of being inexhaustible, were of the same magnitude. But then he looked at the “real” numbers—those marking off the points on a continuous line. Could they, too, be paired up, one to one, with the whole numbers? By a surpassingly clever bit of reasoning, called the “diagonal proof,” Cantor showed that the answer was no. In other words, there were at least two distinct infinities, that of the whole numbers and that of the continuum, and the second was greater than the first.

But was that the end of it? In searching for a still larger species of infinity, Cantor looked to higher dimensions. Surely, he thought, there must be more points in a two-dimensional plane than on a one-dimensional line. For a couple of years, he struggled to prove that the points in a plane could not be paired off one to one with the points in a line, only to find, in 1878, that such a correspondence was indeed possible. A simple trick showed that there were exactly as many points on a one-inch line as there were in all of space. “I see it, but I do not believe it!” Cantor wrote to a colleague.

With the discovery that neither size nor dimension was the way to higher infinities, the quest seemed to stall. But after more than a decade of intense work (interrupted by a stay in a sanatorium after a nervous breakdown), Cantor arrived at a powerful new principle that enabled him to resume the ascent: there are always more sets of things than things. This is obvious enough in a finite world. If you have, say, three objects, you can form eight different sets out of them (including, of course, the empty set). Cantor’s genius lay in extending the same principle into the realm of the infinite.

To make matters a little less abstract, you might pretend that we live in a world with an infinite number of people. Now consider all the possible clubs (sets of people) that might exist in this world. The least exclusive of these clubs—the universal club—will be the one of which absolutely everyone is a member. The most exclusive—the null club—will be the one that has no members at all. In between these two extremes will lie an infinity of other clubs, some with lots of members, some with few. How big is this infinity? Is there any way of matching up clubs and people one to one, thereby showing that the two infinite collections are the same size? Suppose that every person can be paired off with precisely one club, and vice versa. Some of the people will happen to be members of the club with which they are paired (for example, the person who gets paired with the universal club). Others will happen not to be members of their associated club (for example, the person who gets paired with the null club). Those people make up a group you might call the Groucho Club. The Groucho Club is a sort of salon des refusés: it consists of all the people who are paired with clubs that won’t have them as members.

Now, here is where things get interesting. Since the pairing of people with clubs is assumed to be complete, there must be some fellow who is paired up with the Groucho Club. Is that fellow a member of the Groucho Club or not? Well, suppose he is. That means that he must be excluded from the club he is paired with. So he is not a member of the Groucho Club. But if he is not a member of the Groucho Club, then, since the club he is paired with won’t have him, he is a member of the Groucho Club. No matter which way we turn, we get a contradiction. How did we get to this impasse? By supposing that people could be paired off, one to one, with clubs. So that supposition has to be false. Which is to say, the infinity of sets of things is greater than the infinity of things.

The beauty of this principle, which has come to be known as Cantor’s theorem, is that it can be applied over and over again. Given any infinite set, you can always come up with a larger infinity by considering its “power set”—the set of all its subsets. Atop a simple reductio ad absurdum, Cantor built a never-ending tower of infinities. It seemed a dream vision, like Coleridge’s “Kubla Khan.” Yet mathematicians found in this new theory the resources needed to put their subject on a sure footing. “No one shall expel us from the paradise which Cantor has created for us,” one of them declared. Others, however, dismissed Cantor’s infinity of infinities as a “fog on a fog” and “mathematical insanity.” Cantor felt persecuted by these critics, which worsened his nervous condition (he seems to have suffered from a bipolar disorder). Between frequent breakdowns and hospitalizations, he pondered the theological implications of the infinite and, with equal ardor, pursued the theory that Bacon wrote the works of Shakespeare.

Cantor’s theory constitutes “direct evidence that actually-infinite sets can be understood and manipulated, truly handled by the human intellect,” Wallace writes. What makes this achievement so heroic, he observes, is the awful abstractness of infinity: “It’s sort of the ultimate in drawing away from actual experience,” a negation of “the single most ubiquitous and oppressive feature of the concrete world—namely, that everything ends, is limited, passes away.” Wallace is alive to the “dreads and dangers” of abstract thinking. For two millennia, the idea of infinity was thought to pose hazards to one’s sanity. It was Cantor, for all his madness, who managed to tame infinity, and showed that it could be reasoned about without derangement.

Writing accessibly about abstract mathematical ideas poses hazards of its own. One pitfall of such attempts is purple prose. A widely read book about the calculus tells us that “the Cartesian plane is suffused with
a strange and somber silence”; a book about zero says that this number is “a shadow in the slanting light of fear.” Another pitfall is mysticism. In “The Mystery of the Aleph,” a book about infinity, Amir D. Aczel tries to make Georg Cantor into a Kabbalist who entered “God’s secret garden” and lost his sanity in punishment. Rudy Rucker’s “Infinity and the Mind,” a terrific study with real mathematical depth, takes an unwelcome excursion into Zen Buddhism. On the other hand, a little classic called “Playing with Infinity,” by Rozsa Peter, a Hungarian logician who died in 1977, achieves both charm and clarity without a bit of extraneous guff. But all these popularizers, whatever their lapses, conscientiously did the brutal work that is required to make abstract ideas clear, even beautiful, to the beginner. By simplifying, leaving things out, they produced a first approximation to real understanding.

Wallace’s book, by contrast, can’t quite be described as popularization. Wallace assures us that it is “a piece of pop technical writing,” and that his own math background doesn’t go much beyond high school. And yet he has refused to make the usual compromises. “Everything and More” is sometimes as dense as a math textbook, though rather more chaotic. I have never come across a popular book about infinity that packs so much technical detail—especially one that purports to be “compact.” (What Wallace calls a “booklet” actually runs to more than three hundred pages.) The motive behind this is admirable: Wallace is determined to improve on “certain recent pop books that give such shallow and reductive accounts of Cantor’s proofs . . . that the math is distorted and its beauty obscured.” But when a writer’s grasp of his material is less than sure, he risks sacrificing clarity to spectacle, flashing equations and technical terms at his audience like a magician’s deck of cards. Wallace invites us simply to “revel” in the “symbology.” He tells us that some of the terms—“inaccessible ordinals,” “transfinite recursion”—are “fun to say even if one has no clear idea what they’re supposed to denote.” An aesthetic fondness for the math textbook’s visual display may also explain his penchant for initials and abbreviations (“with respect to” becomes “w/r/t,” Galileo becomes “G.G.,” and the “Divine Brotherhood of Pythagoras” is “D.B.P.”), space-saving measures that are hard to square with his energetic volubility.

Still, Wallace’s enthusiasm for the theory of infinity is evident on every page (not least in his conviction that Cantor is “the most important mathematician of the nineteenth century,” a view that few mathematicians or intellectual historians would agree with). And if he is sometimes over his head it is because he has chosen to wade through the deepest waters. The question is whether he will bring his readers along. By way of assistance, Wallace offers “emergency glossaries,” elaborate directions (“end q.e.i. return to para. 2 of (b), in progress”), and frequent warnings and apologies (“Parts of E.G.II are going to be brutal . . . regrets are hereby conveyed”), yet the result is to complicate matters further. In explaining how Cantor first came to consider infinite sets, Wallace brings up Fourier integrals, a topic suitable for an advanced course in mathematics, reassuring readers that all one needs to know about them is that “they’re special kinds of ‘closed-form’ solutions to partial differential equations.” Disarmingly, he provides, at the end of his book, a list of “Quoted and/or Cribbed Material”; but too much of the technical apparatus seems to have arrived by airlift. You can’t stuff something like Riemannian geometry into a footnote, even if you are a master of the form.

Then for whom is the book intended? If laymen will be dazed by its resolute obscurity, expert readers, who tend to be a fussy lot, may take a dim view of the autodidact’s small missteps. Wallace says that a certain curve is “discontinuous” at a point where in fact it is not even defined. He muddles the Law of the Excluded Middle (every proposition is either true or false) by adding an extra clause (every proposition is either true or, if not true, false) that serves no logical purpose. He gives an erroneous account of the work of Kurt Gödel, claiming that the logician showed that “a formal system can’t be both Complete and Consistent.” (That’s the case when the formal system contains arithmetic, but elementary geometry is a formal system that has been proved both complete and consistent.) Nor will philosophers have confidence in an account venturing that one of Cantor’s contemporaries is a “Platonist” (Platonism here being the view that mathematical entities, such as numbers, are timeless and not products of our thought), only to quote him as saying that “numbers are free creations of the human mind”—which is precisely the opposite of Platonism.

A book that prizes difficulty but not rigor is probably not meant for those in search of mathematical illumination; what it offers, in the end, is a purely literary experience. Regarding the nature of that experience, you may find a clue in Ludwig Wittgenstein’s response to Cantor’s great contribution. The thrill we get from discovering that some infinities are bigger than others, Wittgenstein thought, is just a “schoolboy pleasure.” There is nothing awesome about the theory; it does not describe a world of timeless, transcendent, scarcely conceivable entities—it is really no more than a collection of (finite) tricks of reasoning. One might imagine, Wittgenstein said, that the theory of infinite sets was “created by a satirist as a kind of parody of mathematics.” Wallace, whose satiric gifts have long been appreciated, may have achieved something considerable after all—a sly send-up of pop technical writing. “A parody of mathematics”: As a description of Cantor’s work on infinity, it is surely unjust. As a description of Wallace’s, it may be taken as a tribute.

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